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Fixed point results for set-contractions on metric spaces with a directed graph

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Abstract
In this paper, we establish the existence of fixed points for set-valued mappings satisfying certain graph contractions with set-valued domain endowed with a graph. These results unify, generalize, and complement various known comparable results in the literature.
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1 Introduction and preliminaries

Existence of fixed points in ordered metric spaces has been studied by Ran and Reurings [1]. Recently, many researchers have obtained fixed point results for single- and set-valued mappings defined on partially ordered metrics spaces (see, *e.g.*, [2–6]). Jachymski and Jozwik [7] introduced a new approach in metric fixed point theory by replacing the order structure with a graph structure on a metric space. In this way, the results proved in ordered metric spaces are generalized (see also [8] and the references therein); in fact, in 2010, Gwozdz-Lukawska and Jachymski [9], developed the Hutchinson-Barnsley theory for finite families of mappings on a metric space endowed with a directed graph. Abbas and Nazir [10] obtained some fixed point results for power graph contraction pair endowed with a graph. Bojor [11] proved fixed point theorem of φ-contraction mapping on a metric space endowed with a graph. Recently, Bojor [12] proved fixed point theorems for Reich type contractions on metric spaces with a graph. For more results in this direction, we refer to [13–17] and the references mentioned therein. The reader interested in fixed point results of partial metric spaces is referred to [2, 10, 18]. In this paper, we prove fixed point results for set-valued maps, defined on the family of closed and bounded subsets of a metric space endowed with a graph and satisfying graph ϕ-contractive conditions. These results extend and strengthen various known results in [7, 8, 11, 19–21].

Consistent with Jachymski [8], let (X,d) be a metric space and Δ denotes the diagonal of $X \times X$. Let G be a directed graph, such that the set $V(G)$ of its vertices coincides with X and $E(G)$ be the set of edges of the graph which contains all loops, that is, $\Delta \subseteq E(G)$. Also assume that the graph G has no parallel edges and, thus, one can identify G with the pair $(V(G),E(G))$.

Definition 1.1 [8] An operator $f : X \to X$ is called a Banach G-contraction or simply a G-contraction if

(a) f preserves edges of G; for each $x, y \in X$ with $(x, y) \in E(G)$, we have $(f(x), f(y)) \in E(G)$,

(b) f decreases weights of edges of G; there exists $\alpha \in (0,1)$ such that for all $x, y \in X$ with $(x, y) \in E(G)$, we have $d(f(x), f(y)) \leq \alpha d(x, y)$.

If x and y are vertices of G, then a path in G from x to y of length $k \in \mathbb{N}$ is a finite sequence $\{x_n\}$ ($n \in \{0,1,2,\dots,k\}$) of vertices such that $x_0 = x$, $x_k = y$, and $(x_{i-1}, x_i) \in E(G)$ for $i \in \{1,2,\dots,k\}$.

Notice that a graph G is connected if there is a directed path between any two vertices and it is weakly connected if $\widetilde{G}$ is connected, where $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of the edges. Denote by G^{-1} the graph obtained from G by reversing the direction of the edges. Thus,

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

It is more convenient to treat $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric; under this convention, we have

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}).$$

If G is such that $E(G)$ is symmetric, then for $x \in V(G)$, the symbol $[x]_G$ denotes the equivalence class of the relation R defined on $V(G)$ by the rule:

yRz if there is a path in G from y to z.

Recall that if $f : X \to X$ is an operator, then by F_f we denote the set of all fixed points of f. We set also

$$X_f := \{x \in X : (x, f(x)) \in E(G)\}.$$

Jachymski and Jozwik [7] used the following property:

(P) for any sequence $\{x_n\}$ in X, if $x_n \to x$ as $n \to \infty$ and $(x_n, x_{n+1}) \in E(G)$, then $(x_n, x) \in E(G)$.

Theorem 1.2 [7] *Let (X, d) be a complete metric space and let G be a directed graph such that $V(G) = X$. Let $E(G)$ and the triplet (X, d, G) have property* (P). *Let $f : X \to X$ be a G-contraction. Then the following statements hold*:

(1) *$F_f \neq \emptyset$ if and only if $X_f \neq \emptyset$*;

(2) *if $X_f \neq \emptyset$ and G is weakly connected, then f is a Picard operator, i.e., $F_f = \{x^*\}$ and sequence $\{f^n(x)\} \to x^*$ as $n \to \infty$, for all $x \in X$*;

(3) *for any $x \in X_f$, $f|_{[x]_{\widetilde{G}}}$ is a Picard operator*;

(4) *if $X_f \subseteq E(G)$, then f is a weakly Picard operator, i.e., $F_f \neq \emptyset$ and, for each $x \in X$, we have sequence $\{f^n(x)\} \to x^*(x) \in F_f$ as $n \to \infty$.*

For a detailed discussion concerning Picard and weakly Picard operators, we refer to Rus [22, 23] and to Berinde [24, 25].

Let (X,d) be a metric space and let $CB(X)$ be the class of all nonempty closed and bounded subsets of X. For $A,B \in CB(X)$, let

$$H(A,B) = \max\left\{\sup_{b\in B} d(b,A), \sup_{a\in A} d(a,B)\right\},$$

where $d(x,B) = \inf\{d(x,b) : b \in B\}$ is the distance of a point x to the set B. The mapping H is said to be the Pompeiu-Hausdorff metric induced by d.

Throughout this paper, we assume that a directed graph G has no parallel edge and G is a weighted graph in the sense that each vertex x is assigned the weight $d(x,x) = 0$ and each edge (x,y) is assigned the weight $d(x,y)$. Since d is a metric on X, the weight assigned to each vertex x to vertex y need not be zero and, whenever a zero weight is assigned to some edge (x,y), it reduces to a loop (x,x) having weight 0. Further, in Pompeiu-Hausdorff metric induced by metric d, the Pompeiu-Hausdorff weight assigned to each $U, V \in CB(X)$ need not be zero (that is, $H(U,V) \neq 0$) and, whenever a zero Pompeiu-Hausdorff weight is assigned to some $U, V \in CB(X)$, it reduces to $U = V$.

Definition 1.3 Let A and B be two nonempty subsets of X. Now we treat some terminology:

(a) by 'there is an edge between A and B', we mean there is an edge between some $a \in A$ and $b \in B$ which we denote by $(A,B) \subset E(G)$.

(b) by 'there is a path between A and B', we mean that there is a path between some $a \in A$ and $b \in B$.

In $CB(X)$, we define a relation R in the following way:

For $A,B \in CB(X)$, we have ARB if and only if there is a path between A and B.

We say that the relation R on $CB(X)$ is transitive if there is a path between A and B, and there is a path between B and C, then there is a path between A and C.

For $A \in CB(X)$, the equivalence class of A induced by R is denoted by

$$[A]_G = \{B \in CB(X) : ARB\}.$$

Now we consider the mapping $T : CB(X) \to CB(X)$ instead of $T : X \to X$ or $T : X \to CB(X)$ to study fixed points of graph contraction mappings.

For a mapping $T : CB(X) \to CB(X)$, we define the following set:

$$X_T := \{U \in CB(X) : (U, T(U)) \subseteq E(G)\}.$$

Definition 1.4 Let $T : CB(X) \to CB(X)$ be a set-valued mapping. The mapping T is said to be a graph ϕ-contraction if the following conditions hold:

(i) There is an edge between A and B implies there is an edge between $T(A)$ and $T(B)$ for all $A,B \in CB(X)$.

(ii) There is a path between A and B implies there is a path between $T(A)$ and $T(B)$ for all $A,B \in CB(X)$.

(iii) There exists an upper semi-continuous and nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(t) < t$ for each $t > 0$ such that there is an edge between A and B implies

$$H\big(T(A), T(B)\big) \leq \phi\big(H(A,B)\big) \quad \text{for all } A, B \in CB(X). \tag{1.1}$$

Example 1.5

(1) Any constant mapping $T : CB(X) \to CB(X)$ is a graph ϕ-contraction for $\Delta \subset E(G)$.
(2) Any graph ϕ-contraction map for a graph G is also a graph ϕ-contraction for graph G_0, where the graph G_0 is defined by $E(G_0) = X \times X$.

It is obvious if $T : CB(X) \to CB(X)$ is a graph ϕ-contraction for graph G, then T is also graph ϕ-contraction for the graphs G^{-1} and $\widetilde{G}$.

A graph G is said to have property:

(P*) if for any sequence $\{X_n\}$ in $CB(X)$ with $X_n \to X$ as $n \to \infty$, there exists an edge between X_n and X_{n+1} for $n \in \mathbb{N}$, implies that there is a subsequence $\{X_{n_k}\}$ of $\{X_n\}$ with an edge between X_{n_k} and X for $n \in \mathbb{N}$.

Definition 1.6 Let $T : CB(X) \to CB(X)$. The set $A \in CB(X)$ is said to be a fixed point of T if $T(A) = A$. The set of all fixed points of T is denoted by $F(T)$.

A subset Γ of $CB(X)$ is said to be complete if for any set $X, Y \in \Gamma$, there is an edge between X and Y.

Definition 1.7 [19] A metric space (X, d) is called an ε-chainable metric space for some $\varepsilon > 0$ if for given $x, y \in X$, there is $n \in \mathbb{N}$ and a sequence $\{x_n\}$ such that

$$x_0 = x, \qquad x_n = y \quad \text{and} \quad d(x_{i-1}, x_i) < \varepsilon \quad \text{for } i = 1, \ldots, n.$$

We need of the following lemma of Nadler [21] (see also [26]).

Lemma 1.8 *If $U, V \in CB(X)$ with $H(U, V) < \varepsilon$, then for each $u \in U$ there exists an element $v \in V$ such that $d(u, v) < \varepsilon$.*

2 Fixed point results

In this section, we obtain several fixed point results for set-valued selfmaps on $CB(X)$ satisfying certain graph contraction conditions.

Theorem 2.1 *Let (X, d) be a complete metric space endowed with a directed graph G such that $V(G) = X$ and $E(G) \supseteq \Delta$. If $T : CB(X) \to CB(X)$ is a graph ϕ-contraction mapping such that the relation R on $CB(X)$ is transitive, then following statements hold*:

(i) *If $F(T)$ is complete, then the Pompeiu-Hausdorff weight assigned to the $U, V \in F(T)$ is* 0.
(ii) *$X_T \neq \emptyset$ provided that $F(T) \neq \emptyset$.*
(iii) *If $X_T \neq \emptyset$ and the weakly connected graph G satisfies the property* (P*), *then T has a fixed point.*
(iv) *$F(T)$ is complete if and only if $F(T)$ is a singleton.*

Proof To prove (i), let $U, V \in F(T)$. Suppose that the Pompeiu-Hausdorff weight assign to the U and V is not zero. Since T is a graph ϕ-contraction, we have

$$\begin{aligned} H(U,V) &= H\big(T(U),T(V)\big) \\ &\le \phi\big(H(U,V)\big) \\ &< H(U,V), \end{aligned}$$

a contradiction. Hence (i) is proved.

To prove (ii), let $F(T) \neq \emptyset$. Then there exists $U \in CB(X)$ such that $T(U) = U$. Since $\Delta \subseteq E(G)$ and U is nonempty, we conclude that $X_T \neq \emptyset$.

To prove (iii), let $U \in X_T$. As T is a graph ϕ-contraction and $A, B \in CB(X)$, it follows by the hypothesis $CB(X) \subseteq [A]_{\widetilde{G}} = P(X)$, where $P(X)$ denotes the power set of X and so, $T(A) \in [A]_{\widetilde{G}}$. Now for $A \in CB(X)$ and $B \in [A]_{\widetilde{G}}$, there exists a path $\{x_i\}_{i=0}^{n}$ from some $x \in A$ and to $y \in T(A)$, that is, $x_0 = x$ and $x_n = y$ and $(x_{i-1}, x_i) \in E(\widetilde{G})$, for $i = 1, 2, \ldots, n$, such that $x_0 \in A_0 = A$, $x_1 \in A_1, \ldots, x_n \in A_n = T(A)$, where each $A_i \in CB(X)$. Since T is also a graph ϕ-contraction for graph $\widetilde{G}$, for $i = 1, 2, \ldots, n$, we have

$$\begin{aligned} &H\big(T(A_{i-1}),T(A_i)\big) \le \phi\big(H(A_{i-1},A_i)\big), \\ &H\big(T(A_{i-2}),T(A_{i-1})\big) \le \phi\big(H(A_{i-2},A_{i-1})\big), \\ &\cdots \\ &H\big(T(A_0),T(A_1)\big) \le \phi\big(H(A_0,A_1)\big), \end{aligned}$$

and so we obtain

$$H\big(T^n(A),T^{n+1}(A)\big) \le \phi^n\big(H\big(A,T(A)\big)\big)$$

for all $n \in \mathbb{N}$. Now for $m, n \in \mathbb{N}$ with $m > n$,

$$\begin{aligned} H\big(T^n(A),T^m(A)\big) &\le H\big(T^n(A),T^{n+1}(A)\big) + H\big(T^{n+1}(A),T^{n+2}(A)\big) + \cdots \\ &\quad + H\big(T^{m-1}(A),T^m(A)\big) \\ &\le \phi^n\big(H\big(A,T(A)\big)\big) + \phi^{n+1}\big(H\big(A,T(A)\big)\big) + \cdots \\ &\quad + \phi^{m-1}\big(H\big(A,T(A)\big)\big). \end{aligned}$$

On taking the upper limit as $n, m \to \infty$, we get $H(T^n(A), T^m(A))$ converges to 0. Since (X, d) is complete, we have $T^n(A) \to U^*$ as $n \to \infty$ for some $U^* \in CB(X)$. There exists an edge between U and $T(U)$, the fact that T is a graph ϕ-contraction yields the result that there is an edge between $T^n(U)$ and $T^{n+1}(U)$ for all $n \in \mathbb{N}$. By property (P^*), there exists a subsequence $\{T^{n_k}(U)\}$ such that there is an edge between $T^{n_k}(U)$ and U^* for every $n \in \mathbb{N}$. By the transitivity of the relation R, there is a path in G (and hence also in $\widetilde{G}$) between U and U^*. Thus $U \in [U]_{\widetilde{G}}$. Now

$$H\big(T^{n_k+1}(U),T\big(U^*\big)\big) \le \phi\big(H\big(T^{n_k}(U),U^*\big)\big).$$

Now $T^{n_k}(U) \to U^*$ as $n \to \infty$ implies, on taking the upper limit as $n \to \infty$, $T^{n_k+1}(U) \to T(U^*)$ as $n \to \infty$. Thus we obtain $U^* = T(U^*)$.

Finally to prove (iv), suppose the set $F(T)$ is complete. We are to show that $F(T)$ is singleton. Assume to the contrary that there exist $U, V \in CB(X)$ such that $U, V \in F(T)$ and $U \neq V$. By completeness of $F(T)$, there exists an edge between U and V. As T is a graph ϕ-contraction, so we have

$$\begin{aligned} 0 &< H(U,V) \\ &= H\bigl(T(U),T(V)\bigr) \\ &\leq \phi\bigl(H(U,V)\bigr), \end{aligned}$$

a contradiction. Hence $U = V$.

Conversely, if $F(T)$ is singleton, then obviously $F(T)$ is complete. □

The following corollary is a direct consequence of Theorem 2.1(iii).

Corollary 2.2 *Let (X,d) be a complete metric space endowed with a directed graph G such that $V(G) = X$ and $E(G) \supseteq \Delta$. If G is weakly connected, then graph ϕ-contraction mapping $T : CB(X) \to CB(X)$ with $(A_0, A_1) \subset E(G)$ for some $A_1 \in T(A_0)$, has a fixed point.*

Corollary 2.3 *Let (X,d) be a ε-chainable complete metric space for some $\varepsilon > 0$, $T : CB(X) \to CB(X)$ and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be an upper semi-continuous and nondecreasing function with $\phi(t) < t$ for each $t > 0$ with*

$$0 < H(A,B) < \varepsilon.$$

If

$$H\bigl(T(A),T(B)\bigr) \leq \phi\bigl(H(A,B)\bigr) \quad \textit{for all } A,B \in CB(X),$$

then T has a fixed point.

Proof By Lemma 1.8, from $H(A,B) < \varepsilon$, we have for each $a \in A$, an element $b \in B$ such that $d(a,b) < \varepsilon$. Consider the graph G as $V(G) = X$ and

$$E(G) = \bigl\{(a,b) \in X \times X : 0 < d(a,b) < \varepsilon\bigr\}.$$

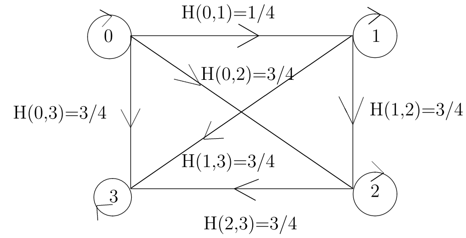
Then the ε-chainability of (X,d) implies that G is connected. For $(A,B) \subset E(G)$, we have from the hypothesis

$$H\bigl(T(A),T(B)\bigr) < \phi\bigl(H(A,B)\bigr).$$

This implies that T is a graph ϕ-contraction mapping.

Also, G has property (P^*). Indeed, if $\{X_n\}$ in $CB(X)$ with $X_n \to X$ as $n \to \infty$ and $(X_n, X_{n+1}) \subset E(G)$ for $n \in \mathbb{N}$, implies that there is a subsequence $\{X_{n_k}\}$ of $\{X_n\}$ such that $(X_{n_k}, X) \subset E(G)$ for $n \in \mathbb{N}$. So by Theorem 2.1(iii), T has a fixed point. □

Figure 1 Pompeiu-Hausdorff weighted graph.



Example 2.4 Let $X = \{0, 1, 2, \dots, n-1\} = V(G)$ and

$$\begin{aligned} E(G) = \bigl\{&(0,0), (1,1), (2,2), \dots, (n-1, n-1), \\ &(0,1), (0,2), \dots, (0, n-1), \\ &(1,2), (1,3), \dots, (1, n-1), \\ &\dots \\ &(n-2, n-1)\bigr\}. \end{aligned}$$

Let $V(G)$ be endowed with metric $d : X \times X \to \mathbb{R}^+$ defined by

$$\begin{aligned} &d(0,0) = d(1,1) = \cdots = d(n-1, n-1) = 0, \\ &d(0,1) = d(1,0) = \frac{1}{n}, \\ &d(0,2) = d(2,0) = d(1,2) = d(2,1) = \cdots = d(n-2, n-1) = d(n-1, n-2) = \frac{n}{n+1}. \end{aligned}$$

The Pompeiu-Hausdorff weights (for $n = 4$) assigned to $A, B \in CB(X)$ are shown in Figure 1.

Furthermore,

$$H(A,B) = \begin{cases} \frac{1}{n}, & \text{if } A, B \subseteq \{0,1\} \text{ with } A \neq B, \\ \frac{n}{n+1}, & \text{if } A \text{ or } B \text{ (or both)} \nsubseteq \{0,1\} \text{ with } A \neq B, \\ 0, & \text{if } A = B. \end{cases}$$

Define $T : CB(X) \to CB(X)$ as follows:

$$T(U) = \begin{cases} \{0\}, & \text{if } U \subseteq \{0,1\}, \\ \{0,1\}, & \text{if } U \subsetneq \{0,1\}. \end{cases}$$

Note that, for all $A, B \in CB(X)$ with edge between A and B, there is an edge between $T(A)$ and $T(B)$. Also there is a path between A and B implies that there is a path between $T(A)$ and $T(B)$.

Define $\phi : [0, \infty) \to [0, \infty)$ by

$$\phi(t) = \begin{cases} \frac{4t}{5}, & \text{if } t \in [0, \frac{5}{2}), \\ \frac{2^n(2^{n+1}t-3)}{2^{2n+1}-1}, & \text{if } t \in [\frac{2^{2n}+1}{2^n}, \frac{2^{2(n+1)}+1}{2^{n+1}}], n \in \mathbb{N}. \end{cases}$$

An easy computation shows that ϕ is continuous on $[0, \infty)$ and $\phi(t) < t$ for all $t > 0$.

Now for all $A, B \in CB(X)$, we consider the following cases:

(a) For $A, B \subseteq \{0,1\}$, we have $H(T(A), T(B)) = 0$.

(b) If $A \subseteq \{\{0\},\{1\},\{0,1\}\}$ and $B \subsetneq \{\{0\},\{1\},\{0,1\}\}$, then we have

$$H\big(T(A), T(B)\big) = H\big(\{0\},\{0,1\}\big) = \frac{1}{n} < \frac{4n}{5n+5} = \phi\left(\frac{n}{n+1}\right) = \phi\big(H(A,B)\big).$$

(c) In the case $A, B \subsetneq \{\{0\},\{1\},\{0,1\}\}$, we have

$$H\big(T(A), T(B)\big) = H\big(\{0,1\},\{0,1\}\big) = 0.$$

Obviously, (1.1) is satisfied in the cases (a), (b), and (c).

Hence for all $A, B \in CB(X)$ having an edge between A and B, (1.1) is satisfied and so T is a graph ϕ-contraction. Thus all the conditions of Theorem 2.1 are satisfied. Moreover, {0} is the fixed point of T and $F(T)$ is complete.

Remark 2.5

(1) If $E(G) := X \times X$, then clearly G is connected and Theorem 2.1 improves and generalizes Theorem 2.5 in [19], Theorems 2.1-2.3 in [11] and Theorem 3.1 in [7].

(2) Theorem 2.1 with the graph G improves and generalizes Theorem 2.1 in [20] from single valued to set-valued mappings.

(3) If $E(G) := X \times X$, then clearly G is connected and our Corollary 2.2 extends and generalizes Theorem 2.5 in [19], Theorem 3.2 in [21], and Theorem 3.1 in [7].

(4) If $E(G) := X \times X$, then clearly G is connected and our Corollary 2.3 improves and generalizes Theorem 3.2 in [21] and Theorem 3.1 in [7].

(5) Corollary 2.3 extends and improves the Banach contraction theorem and Theorem 5.1 in [27].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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